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Evolution Equations with Infinite Delay

To the Memory of Professor T. Yoshizawa

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1 Introduction

Suppose that A is the infinitesimal generator of a C_0 semigroup $T(t)$ on a Banach space E with norm $|\cdot|$. We consider the evolution equation with infinite delay such that

$$u'(t) = Au(t) + L(u_t), \tag{1.1}$$

where u_t is a function mapping $(-\infty, 0]$ into E defined by $u_t(\theta) = u(t+\theta)$ for $\theta \in (-\infty, 0]$. The operator L is a bounded linear operator on the phase space $\mathcal{B}$ into E.

$\mathcal{B}$ is a Banach space of some functions mapping $(-\infty, 0]$ into E. The norm in $\mathcal{B}$ is denoted by $\|\cdot\|$. For a complex number λ and for $x \in E$ we define a function $\varepsilon_\lambda \otimes x : (-\infty, 0] \to E$ by $(\varepsilon_\lambda \otimes x)(\theta) = e^{\lambda\theta}x, \quad \theta \in (-\infty, 0]$. We assume the following axioms on $\mathcal{B}$:

(H-1) There exists a constant H such that $|\phi(0)| \leq H\|\phi\|$ for every $\phi \in \mathcal{B}$.

(H-2) If a function $u : (-\infty, \sigma + a) \to E$ is continuous on $[\sigma, \sigma + a)$, and if $u_\sigma \in \mathcal{B}$, then

(i) $u_t \in \mathcal{B}$ for all $t \in [\sigma, \sigma + a)$ and u_t is a $\mathcal{B}$ valued continuous function of $t \in [\sigma, \sigma + a)$,

(ii) $\|u_t\| \leq K(t-\sigma)\sup\{|u(s)| : \sigma \leq s \leq t\} + M(t-\sigma)\|u_\sigma\|$

for all $t \in [\sigma, \sigma + a)$, where $K(r), M(r), r \geq 0$, are nonnegative, measurable, locally bounded functions which are independent of u.

(H-3) If $\{\phi^n\}$ is a Cauchy sequence in $\mathcal{B}$, and if the sequence $\{\phi^n(\theta)\}$ converges to a function $\phi(\theta)$ uniformly on every compact interval of $(-\infty, 0]$, then ϕ lies in $\mathcal{B}$ and $\lim_{n\to\infty} \|\phi^n - \phi\| = 0$.

(H-4) There exists a constant γ such that $\varepsilon_\lambda \otimes x \in \mathcal{B}$ for $\Re\lambda > \gamma$ and $x \in E$, and that

$$\|\varepsilon_\lambda\| := \sup\{\|\varepsilon_\lambda \otimes x\| : x \in E,\ |x| \leq 1\}$$

is finite for each λ with $\Re\lambda > \gamma$, and bounded for $\Re\lambda > \gamma_1$ for some $\gamma_1 \geq \gamma$.

We call the constant γ in (H-4) the abscissa of the exponent of the space $\mathcal{B}$. The hypothesis (H-3) implies that the integral in $\mathcal{B}$ is computed from the integral in E in the following manner.

Lemma 1.1 *If $f : [a,b] \to \mathcal{B}$ is a continuous function such that $f(t)(\theta)$ is continuous for $(t,\theta) \in [a,b] \times (-\infty, 0]$, then*

$$\left[\int_a^b f(t)\,dt\right](\theta) = \int_a^b f(t)(\theta)\,dt, \qquad \theta \in (-\infty, 0].$$

The growth bound $\omega_s(T)$, and the essential growth bound $\omega_e(T)$ are defined by

$$\omega_s(T) := \lim_{t\to\infty} \frac{\log\|T(t)\|}{t} = \inf_{t>0} \frac{\log\|T(t)\|}{t},$$

$$\omega_e(T) := \lim_{t\to\infty} \frac{\log\alpha(T(t))}{t} = \inf_{t>0} \frac{\log\alpha(T(t))}{t},$$

where $\alpha(T(t))$ is the measure of noncompactness of $T(t)$ which is introduced by the Kuratowskii measure of noncompactness of bounded sets of X, cf [2]. Then the spectral radius $r_s(T(t))$ and the essential spectral radius $r_e(T(t))$ are given as $r_s(T(t)) = \exp(t\omega_s(T))$ and $r_e(T(t)) = \exp(t\omega_e(T))$. Let A be the infinitesimal generator of $T(t)$, $\sigma(A)$ the spectrum of A, $E_\sigma(A)$ the essential spectrum of A, and set $N_\sigma(A) := \sigma(A) \setminus E_\sigma(A)$. The points in $N_\sigma(A)$ are called normal eigenvalues of A.

The important fact for our works is that the following inequalities hold:

$$\beta_s(A) := \sup\{\Re\lambda : \lambda \in \sigma(A)\} \leq \omega_s(T)$$

$$\beta_e(A) := \sup\{\Re\lambda : \lambda \in E_\sigma(A)\} \leq \omega_e(T).$$

The first inequality is well known. The second inequality is proved in Webb [7], and it implies the following theorems.

Theorem 1.2 *Let $T(t)$ and A be as in the above, and suppose that $\omega_e(T) < \omega_s(T)$. Then the following results hold:*

(i) *There exists at least one point $\lambda \in N_\sigma(A)$ such that $\Re\lambda = \omega_s(T)$: consequently, $\beta_s(A) = \omega_s(T)$ and $N_\sigma(A) \neq \emptyset$.*

(ii) *For any $b, \omega_e(T) < b < \omega_s(T)$, the set $\sigma(A) \cap \{\lambda : \Re\lambda \geq b\}$ consists of finite normal eigenvalues of A, and $\sup\{\Re\lambda : \lambda \in \sigma(A), \Re\lambda < b\} < b$.*

2 Semigroup generated by mild solutions

Let $\phi \in \mathcal{B}$. The strong solution of (1.1) through $(0, \phi)$ is a function $u : (-\infty, \infty) \to E$ which has the following properties: (i) $u_0 = \phi$ and u is continuous, differentiable on $[0, \infty)$, and $u(t) \in D(A)$ for $t \geq 0$; (ii) (1.1) holds for $t \geq 0$. The mild solution of (1.1) through $(0, \phi)$ is a function $u : (-\infty, \infty) \to E$ which has the following properties: (i) $u_0 = \phi$ and u is continuous on $[0, \infty)$;

$$\text{(ii)} \qquad u(t) = T(t)\phi(0) + \int_0^t T(t-s)L(u_s)\, ds, \qquad t \geq 0.$$

By the usual method of successive aproximation, we can prove that, for every $\phi \in \mathcal{B}$, there exists a unique mild solution through $(0, \phi)$; cf. [3],[5],[6], and the references therein.

Denote by $u(t, \phi)$ this mild solution. Define the solution operator $U_L(t)$ on $\mathcal{B}$ by

$$(U_L(t)\phi)(\theta) = u(t + \theta, \phi), \qquad \theta \in (-\infty, 0].$$

Then using the axioms of the phase space, we see that $U_L(t)$ is a C_0 semigroup of bounded linear operators on $\mathcal{B}$. Denote by A_L the infinitesimal generator of $U_L(t)$.

In the particular case that $L \equiv 0$, $U_0(t)$ is given by $(U_0(t)\phi)(\theta) = T(t + \theta)\phi(0)$, $-t < \theta \leq 0$ and $(U_0(t)\phi)(\theta) = \phi(t + \theta)$, $\theta \leq -t$. Set $K_L(t) = U_L(t) - U_0(t)$, which is given by $(K_L(t)\phi)(\theta) = z(t + \theta, \phi)$, $\theta \in (-\infty, 0]$, where

$$z(t,\phi) = \begin{cases} \int_0^t T(t-s)L(u_s)\,ds & t > 0 \\ 0 & t \le 0. \end{cases}$$

Taking the Laplace transform of $U_L(t) = U_0(t) + K_L(t)$, we can compute the resolvent $R(\lambda, A_L)$. To describe the result, we introduce the closed linear operator $\Delta(\lambda)$ as $\Delta(\lambda)x = (\lambda I - A - L_\lambda)x, \quad x \in D(A)$, where $L_\lambda x = L(\varepsilon_\lambda \otimes x)$. It is well defined for $\Re\lambda > \gamma$. If $\lambda \in \rho(A)$, then we can write $(\lambda I - A - L_\lambda) = (I - L_\lambda R(\lambda, A))(\lambda I - A)$. Hence $\Delta(\lambda)^{-1}$ exists as a bounded linear operator on $\mathcal{B}$ as long as $\Re\lambda$ is sufficiently large. Let A_0 be the infinitesimal generator of $U_0(t)$.

Theorem 2.1 *There exists an ω such that, if $\Re\lambda > \omega$, then*

$$R(\lambda, A_L)\phi = R(\lambda, A_0)\phi + \varepsilon_\lambda \otimes \Delta(\lambda)^{-1} L(R(\lambda, A_0)\phi), \qquad \phi \in \mathcal{B}.$$

Since $\phi = R(\lambda, A_L)(\lambda\phi - A_L\phi)$ for $\phi \in D(A_L)$, the equation $A_L\phi = \psi$ holds if and only if $\phi = R(\lambda, A_L)(\lambda\phi - \psi)$. Namely, we can compute A_L itself from the representation of $R(\lambda, A_L)$. To do so, we use the infinitesimal generator B of the trivial C_0 semigroup $S(t)$ on $\mathcal{B}$ defined as $[S(t)\phi](\theta) = \phi(0),\ t + \theta \ge 0$, and $[S(t)\phi](\theta) = \phi(t+\theta),\ t + \theta < 0$.

Theorem 2.2 *A function ϕ lies in $D(A_L)$ if and only if $\phi(0) \in D(A)$ and $\phi - \lambda^{-1}\varepsilon_\lambda \otimes (A\phi(0) + L(\phi)) \in D(B)$ for some $\lambda > \omega$, and*

$$A_L\phi = \varepsilon_\lambda \otimes (A\phi(0) + L(\phi)) + B\left(\phi - \lambda^{-1}\varepsilon_\lambda \otimes (A\phi(0) + L(\phi))\right).$$

In particular, $(A_L\phi)(0) = A\phi(0) + L(\phi)$ for $\phi \in D(A_L)$.

The second equation avobe follows from the fact that $(B\phi)(0) = 0$ for $\phi \in D(B)$. As a result, we have the following theorem.

Theorem 2.3 *If $\phi \in D(A_L)$, the mild solution $u(t,\phi)$ is a strong solution.*

3 Growth bound and compact semigroups

The axiom (H-4) says that, if $\Re\lambda > \gamma$, ε_λ is regarded as an element of $\mathcal{L}(E, \mathcal{B})$, the Banach space of bounded linear operators on E into $\mathcal{B}$. We have the following estimate for the abscissa γ. Let $S_0(t)$ be the restriction of $S(t)$ to the subspace $\mathcal{B}_0 = \{\phi \in \mathcal{B} : \phi(0) = 0\}$.

Theorem 3.1 *If $\Re\lambda > \omega_s(S_0)$, then ε_λ lies in $\mathcal{L}(E, \mathcal{B})$, and it is holomorphic for λ. Hence the abcissa γ in (H-4) satisfies $\gamma \leq \omega_s(S_0)$.*

Theorem 3.2 *For the semigroup $S_0(t)$, the essential growth bound is the same as the growth bound: $\omega_e(S_0) = \omega_s(S_0)$.*

Let $\mathcal{BU}_\gamma$ be the space of continuous functions $\phi : (-\infty, 0] \to E$ such that $e^{-\gamma\theta}\phi(\theta)$ is bounded, uniformly continuous for $\theta \in (-\infty, 0]$, and define the norm in this space as $\|\phi\| = \sup\{e^{-\gamma\theta}|\phi(\theta)| : \theta \in (-\infty, 0]\}$. Then this space satisfies the axioms (H-1,2,3,4), and γ in the definition is the abscissa of the exponent of this space.

Another space for $\mathcal{B}$ is made of the set of measurable functions $\phi : (-\infty, 0] \to E$ such that $e^{-\gamma\theta}|\phi(\theta)|$ is integrable on $(-\infty, 0]$, where the seminorm is defined by

$$\|\phi\| = |\phi(0)| + \int_{-\infty}^{0} e^{-\gamma\theta}|\phi(\theta)|\, d\theta.$$

Denote by $E \times \mathcal{L}_\gamma$ the quotient space with respect to this seminorm. Then this space is a Banach space satisfying (H-1,2,3,4), and γ is the abscissa of the exponent.

Theorem 3.3 *If $\mathcal{B} = \mathcal{BU}_\gamma$ or $E \times \mathcal{L}_\gamma$, then $\omega_e(S_0) = \omega_s(S_0) = \gamma$.*

Suppose that the original semigroup $T(t)$ is a compact semigroup. Then $K_L(t) = U_L(t) - U_0(t)$ is a compact operator for $t > 0$. This implies that $\alpha(U_L(t)) = \alpha(U_0(t))$ for $t > 0$. The following theorem follows from the definition of the essential growth bound.

Theorem 3.4 *If $T(t)$ is a compact semigroup, then $\omega_e(U_L) = \omega_e(U_0)$.*

We have the estimate of $\alpha(U_0(t))$ in terms of $H, K(r), M(r)$ in the axioms of the phase space $\mathcal{B}$, and the constant $\gamma_T = \overline{\lim}_{t\to 0}\|T(t)\|$.

Theorem 3.5 *Let $T(t)$ be a compact semigroup on E. Then the following estimates hold for $t > 0$.*

(i) $\alpha(U_0(t)) \le C_1\overline{\lim}_{s\to t-0}M(s)$, *where* $C_1 = H\overline{\lim}_{\epsilon\to 0}K(\epsilon)\max\{1,\gamma_T\} + \overline{\lim}_{\epsilon\to 0}M(\epsilon)$.

(ii) *Suppose that every bounded, continuous function $\phi : (-\infty, 0] \to E$ lies in $\mathcal{B}$ in the manner that* $\sup\{|\phi(\theta)| : \theta \in (-\infty, 0]\} \le J\|\phi\|$, *where J is a constant independent of ϕ. Then* $\alpha(U_0(t)) \le (1 + JH)C_1\overline{\lim}_{s\to t-0}\alpha(S_0(s))$.

4 An example

Let $E = L^2([0,\pi], C)$, the set of square integrable functions on $[0,\pi]$. Consider the equation

$$u'(t) = Au(t) + b\int_{-\infty}^{t} e^{-c(t-s)}u(s)\,ds, \qquad (4.1)$$

where A is defined as $Af = f''$ for $f \in E$ such that f is continuously differentiable, the derivative f' is abosolutely continuous, $f'' \in E$, and that $f(0) = f(\pi) = 0$. It is well known that A is a closed linear operator with dense domain. It is self adjoint, the spectrum of A consists of only point spectrum $\lambda = -n^2, n = 1, 2, \cdots$, $R(\lambda, A)$ has a pole of order 1 at these points, and $|R(\lambda, A)| \le 1/|\lambda+1|$ for $\Re\lambda > -1$. Hence A is the infinitesimal generator of a C_0 semigroup $T(t)$ such that $|T(t)| \le e^{-t}$ for $t \ge 0$. Furthermore, $T(t)$ is a compact semigroup.

Notice that

$$|L(\phi)| := \left|b\int_{-\infty}^{0} e^{c\theta}\phi(\theta)\,d\theta\right| \le |b|\int_{-\infty}^{0} e^{(c+\gamma)\theta}e^{-\gamma\theta}|\phi(\theta)|\,d\theta.$$

If $\gamma = -c$, we have that $|L(\phi)| \le |b|\|\phi\|$ for $\phi \in E \times \mathcal{L}_{-c}$. If $\gamma > -c$, we have that $|L(\phi)| \le |b|(c+\gamma)^{-1}\|\phi\|$ for $\phi \in \mathcal{BU}_\gamma$. Hence, we can take the space $E \times \mathcal{L}_{-c}$ or $\mathcal{BU}_\gamma, \gamma > -c$, for the phase space of Equation (4.1).

The characteristic operator $\Delta(\lambda)$ now becomes

$$\Delta(\lambda)f = \lambda f - Af - b\int_{-\infty}^{0} e^{(c+\lambda)\theta}\,d\theta f \qquad f \in D(A), \;\; \Re\lambda > \gamma \ge -c.$$

If we set $h(\lambda) = \lambda - b/(c+\lambda)$ for $\Re\lambda > -c$, then we can write $\Delta(\lambda) = h(\lambda)I - A$. Since, from Theorem 3.5 (i), $\omega_e(U_L) \leq \gamma$, the spectrum of A_L in $\Re\lambda > \gamma$ consists of only normal eigenvalues. Let λ be such an eigenvalue. Then from [4] it follows that $N(\Delta(\lambda)) \neq \{0\}$. Thus we see that $h(\lambda) = -n^2$ for some $-n^2 \in P_\sigma(A)$. Set $\Lambda_n = \{\lambda : h(\lambda) = -n^2, \Re\lambda > -c\}$ and $\Lambda = \cup_{n\geq 1}\Lambda_n$.

The equation $h(\lambda) = -n^2$ becomes $(\lambda + c)(\lambda + n^2) = b$, which has the roots $\kappa_n = [-(c+n^2) - \sqrt{D}]/2$, $\lambda_n = [-(c+n^2) + \sqrt{D}]/2$, where $D = (c+n^2)^2 - 4(cn^2 - b)$. We are interested in the roots whose real parts are greater than $-c$.

Proposition 4.1

Case (i): $b > 0$. $\Lambda = \{\lambda_n : n \geq 1\}$, *and* $\lambda_1 > \lambda_2 > \cdots > \lambda_n \to -c \quad (n \to \infty)$.

(i-1): $c \leq 0$. λ_n *are all positive;* (i-2): $c > 0$. *If* $0 < b < c$, λ_n *are all negative. If* $b = n^2 c, n \geq 1$, *then* $\lambda_1 > \lambda_2 > \cdots > \lambda_n = 0 > \lambda_{n+1} > \cdots$. *If* $n^2 c < b < (n+1)^2 c, n \geq 1$, *then* $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0 > \lambda_{n+1} > \cdots$.

Case (ii): $b \leq 0$. Λ *is an empty set or finite set.*

(ii-1): $c \leq 1$. $\Lambda = \emptyset$; (ii-2): $c > 1$. *There exists an integer* n_c *such that* $\Lambda = \{\kappa_n, \lambda_n : 1 \leq n \leq n_c\}$, *and the following cases occur. If* $b < -(c-1)^2/4$, *then* κ_n, λ_n *are all imaginary numbers with real part* $x_n \leq x_1 = -(c+1)/2$. *If* $-(c-1)^2/4 \leq b \leq 0$, *then, for some* $k \leq n_c$, *the first* k *terms of both* $\{\kappa_n\}$ *and* $\{\lambda_n\}$ *are real numbers for which* $\lambda_1 < 0$ *is the maximum number, and the rest terms are imaginary numbers whose real parts are less than* λ_1.

Theorem 4.2 *Suppose that* $\lambda_n \in \Lambda$. *If* $b = 0$ *or* λ_n *is a simple root of* $h(\lambda) = -n^2$, *then* $R(\lambda, A_L)$ *has a pole of order* 1 *at* $\lambda = \lambda_n$. $M_{\lambda_0}(A_L) = N(A_L - \lambda_n I)$ *consists of functions* $\phi(\theta, x) = a_0 e^{\lambda_n\theta} \sin nx$, *where* a_0 *is an arbitrary complex constant.*

If λ_n *is a double root of* $h(\lambda) = -n^2$, *then* $R(\lambda, A_L)$ *has a pole of order* 2 *at* $\lambda = \lambda_n$. $M_{\lambda_0}(A_L) = N((A_L - \lambda_n I)^2)$ *consists of functions* $\phi(\theta, x) = (a_0 + a_1\theta)e^{\lambda_n\theta} \sin nx$, *where* a_0, a_1 *are arbitrary complex constants.*

Proof From the theorem [4], $\phi \in N(A_L - \lambda_n)$ if and only if $\phi = \varepsilon_{\lambda_n} \otimes f$

for some $f \in E$ such that $\Delta(\lambda_n)f = 0$. Recall that

$$\Delta(\lambda_n)f = (\lambda_n I - A - \frac{b}{\lambda_n + c}I)f = (-n^2 - A)f.$$

From the definition of A it follows that $f(x) = a_0 \sin nx, 0 \leq x \leq \pi$.

In the similar manner, $\phi \in N((A_L - \lambda_n)^2)$ if and only if $\phi = \varepsilon_{\lambda_n} \otimes f_0 + \varepsilon'_{\lambda_n} \otimes f_1$ for some f_0, f_1 such that $D_1(\lambda_n)\mathrm{col}[f_0, f_1] = 0$, i.e.

$$\Delta(\lambda_n)f_0 + \Delta'(\lambda_n)f_1 = 0 \qquad \Delta(\lambda_n)f_1 = 0.$$

It is easy to see that $\Delta'(\lambda_n) = (1 + b(\lambda_n + c)^{-2})I$.

Now consider the condition that $h(\lambda) = -n^2$ and $\Delta'(\lambda) = 0$, that is,

$$(\lambda + c)(\lambda + n^2) = b \qquad 1 + b(\lambda + c)^{-2} = 0.$$

From the second equation, $b \neq 0$. Eliminating b, we have that $\lambda + n^2 = -(\lambda + c)$; hence, $\lambda = -(c + n^2)/2 = x_n$, the x coordinate of the minimum point of Γ_n. This means that, $\Delta'(\lambda_n) = 0$ if and only if $b \neq 0$ and λ_n is a double root of $h(\lambda) = -n^2$.

Suppose that $b = 0$ or λ_n is a simple root. From the equation $\Delta(\lambda_n)f_1 = 0$, we have $f_1(x) = a_1 \sin nx$. Set $a = (1 + b(\lambda_n + 1)^{-2})a_1$. Then $a \neq 0$ if and only if $a_1 \neq 0$, and the equation $\Delta(\lambda_n)f_0 + \Delta'(\lambda_n)f_1 = 0$ becomes

$$-n^2 f(x) - f''(x) + a \sin nx = 0 \qquad f(0) = f(\pi) = 0.$$

The solution of the first, differential equation is

$$\begin{aligned} f(x) &= \cos nx \left[c_1 + (2n)^{-1} a \left((2n)^{-1} \sin 2nx - x\right)\right] \\ &+ \sin nx \left(c_0 - (2n)^{-1} a \cos 2nx\right). \end{aligned}$$

It satisfies the boundary condition if and only if $c_1 = a = 0$. Thus we have that $f_1(x) = 0, f_0(x) = c_0 \sin nx$. Thus the function ϕ lies in $N((A_L - \lambda_n I)^2)$ if and only if

$$\phi(\theta, x) = c_0 e^{\lambda_n \theta} \sin nx.$$

This shows that $N(A_L - \lambda_n I)^2) = N(A_L - \lambda_n I)$. Thus the eigenspace is the generalized eigenspace of dimension 1, and $R(\lambda_n, A_L)$ has a simple pole at λ_n.

Suppose that $b \neq 0$ and λ_n is a double root. Then $\Delta(\lambda_n)f_0 = \Delta(\lambda_n)f_1 = 0$; hence $\phi(\theta, x)$ is given as in the theorem . To show that $N((A_L - \lambda_n)^3) = N((A_L - \lambda_n)^2)$, consider the equation $D_2(\lambda_n)\mathrm{col}[f_0, f_1, f_2] = 0$. Since

$$\Delta''(\lambda_n) = -2b(\lambda_n + c)^{-3}I,$$

the equation becomes

$$\Delta(\lambda_n)f_0 + \alpha f_2 = 0 \quad \Delta(\lambda_n)f_1 = 0 \quad \Delta(\lambda_n)f_2 = 0,$$

where $\alpha = -2b(\lambda_n + c)^{-3}$. From the second equation, it follows taht $f_1(x) = a_1 \sin nx$. Since $\alpha \neq 0$, we can apply the result above for the first, and the third equation. As a result, it follows that $f_0(x) = a_0 \sin nx, f_2(x) = 0$. Hence, $N((A_L - \lambda_n)^3) \subset N((A_L - \lambda_n)^2)$, and $N((A_L - \lambda_n I)^2)$ is the generalized eigenspace of dimension 2, and $R(\lambda, A)$ has a pole of order 2 at λ_n.

Define a curve $b = \chi(c)$ in the c-b plane by $\chi(c) = 0,\ c \leq 1; \chi(c) = -(c-1)^2/4,\ c > 1$. Following the Proposition 4.1, we devide c-b plane into the subregions as $\Pi_1 : b > \chi(c), -\infty < c < \infty,\ \Pi_2 : b \leq \chi(c), c > 1,\ \Pi_3 : b \leq \chi(c), c \leq 1$.

Theorem 4.3 *Take the space $\mathcal{B} = \mathcal{BU}_\gamma$. If $\gamma > -c$ is sufficiently close to $-c$, then the growth bound of U_L becomes as follows:*

$$\omega_s(U_L) = \begin{cases} \lambda_1 & \text{if} \quad (c,b) \in \Pi_1 \\ -(c+1)/2 & \text{if} \quad (c,b) \in \Pi_2. \end{cases}$$

If $(c, b) \in \Pi_3$, then $\omega_s(U_L) \leq \gamma$ whenever $\gamma > -c$.

Theorem 4.4 *Take the space $\mathcal{B} = E \times \mathcal{L}_{-c}$. Then the growth bound, and the essential growth bound of U_L becomes as follows:*

$$\omega_e(U_L) = -c, \qquad \omega_s(U_L) = \begin{cases} \lambda_1 & \text{if} \quad (c,b) \in \Pi_1 \\ -(c+1)/2 & \text{if} \quad (c,b) \in \Pi_2 \\ -c & \text{if} \quad (c,b) \in \Pi_3. \end{cases}$$

Theorem 4.5 *If $\mathcal{B} = E \times \mathcal{L}_{-c}$, then the following assertions hold:*

(i) *If $(c, b) \in \Pi_1$ and $b > c$, then $\lambda_1 > 0$, and $\|U_L(t)\| \geq e^{\lambda_1 t}$ for $t \geq 0$;* (ii) *If $(c, b) \in \Pi_1$ and $b = c$, then there exists an M such that $1 \leq \|U_L(t)\| \leq M$*

for $t \geq 0$. *If* $(c,b) \in \Pi_1$ *and* $b < c$, *then* $\lambda_1 < 0$, *and* $\|U_L(t)\| \leq M_\epsilon e^{t(\lambda_1+\epsilon)}$ *for* $t \geq 0$; (iii) *If* $(c,b) \in \Pi_2$, *then* $\|U_L(t)\| \leq M_\epsilon e^{t(-(c+1)/2+\epsilon)}$ *for* $t \geq 0$.

If $\mathcal{B} = \mathcal{BU}_\gamma, \gamma > -c$, *then the assertions above hold provided* γ *is sufficiently close to* $-c$.

Theorem 4.6 *In the case* $(c,b) \in \Pi_3$, *we have different estimates of* $\|U_L(t)\|$ *according to the choice of* $\mathcal{B}$.

Choose $\mathcal{B} = \mathcal{BU}_\gamma, \gamma > -c$. *If* $c > 0$, *then we can take a negative* γ *and* $\|U_L(t)\| \leq M_\epsilon e^{t(\gamma+\epsilon)}$ *for* $t \geq 0$. *If* $c \leq 0$, *then* γ *becomes positive, and we only know that* $\|U_L(t)\| \leq M_\epsilon e^{t(\gamma+\epsilon)}$ *for* $t \geq 0$.

Choose $\mathcal{B} = E \times \mathcal{L}_{-c}$. *If* $c > 0$, *then* $\|U_L(t)\| \leq M_\epsilon e^{t(-c+\epsilon)}$ *for* $t \geq 0$. *If* $c < 0$, *then* $\|U_L(t)\| \geq e^{-ct}$ *for* $t \geq 0$. *If* $c = 0, b \leq 0$, *then* $\|U_L(t)\| \geq 1$.

Corollary 4.7 *Take the phase space as* $\mathcal{B} = \mathcal{BU}_\gamma, \gamma > -c$. *If* γ *is sufficiently close to* $-c$, *then the null solution of Equation (4.1) has the following stability: if* $b = c > 0$, *it is stable but not asmptotically stable; if* $c > 0, c > b$, *it is exponentially asymptotically stable. If* $(c,b) \in \Pi_1, b > c$, *then the null solution of Equation (4.1) is not stable for any choice of* $\gamma > -c$.

Corollary 4.8 *Take the phase space as* $\mathcal{B} = E \times \mathcal{L}_{-\gamma}$. *The null solution of Equation (4.1) is exponentially asymptotically stable if and only if* $c > 0$ *and* $b < c$. *If* $c > 0$ *and* $b = c$, *it is stable but not asymptotically stable. If* $c < 0$, *or if* $c \geq 0$ *and* $b > c$, *then it is not stable.*

In the case $c = 0$ the equation becomes

$$u'(t) = Au(t) + b\int_{-\infty}^{0} u(t+\theta)d\theta.$$

Since $|T(t)f| \leq e^{-t}|f|$ for $t \geq 0, f \in E$, it follows that

$$\begin{aligned}
\|U_0(t)\phi\| &= |T(t)\phi(0)| + \int_{-t}^{0} |T(t+\theta)\phi(0)|\, d\theta + \int_{-\infty}^{-t} |\phi(t+\theta)|\, d\theta \\
&\leq e^{-t}|\phi(0)| + \int_{0}^{t} e^{-s}|\phi(0)|\, ds + \int_{-\infty}^{0} |\phi(s)|\, ds \\
&= |\phi(0)| + \int_{-\infty}^{0} |\phi(\theta)| d\theta = \|\phi\|.
\end{aligned}$$

Hence, we have that $\|U_0(t)\| \leq 1$ for $t \geq 1$. Since $\|U_0(t)\| \geq \alpha(U_0(t)) \geq 1$, it follows that $\|U_0(t)\| = \alpha(U_0(t)) \equiv 1$ in the space $E \times \mathcal{L}_0$.

If $b = 0$, then $U_L(t) = U_0(t)$, and the null solution is stable. If $b > 0$, the null solution is not stable from Corollary 4.8. If $b < 0$, from Theorem 4.6 we only know that $\|U_L(t)\| \geq 1$. Is the null solution stable or not? About this interesting problem, Murakami has informed us of the following stability result.

Theorem 4.9 *If $c = 0, b < 0$, the null solution of Equation (4.1) is $\mathcal{B} - E$ uniformly asymptotically stable: that is, there exists a constant M such that $|u(t,\phi)| \leq M\|\phi\|$ for $t \geq 0, \phi \in \mathcal{B}$, and for any $\epsilon > 0$ there exist a $\tau(\epsilon) > 0$ such that $|u(t,\phi)| \leq \epsilon\|\phi\|$ for $t > \tau(\epsilon)$, $\phi \in \mathcal{B}$.*

Proof. Let $\{f_n\}, n = 1, 2, \cdots$, be the complete orthonormal system of the self adjoint operator A, and set $u^n(t) =< u(t), f^n >$. Then $u(t) = \sum_{n\geq 1} u^n(t) f^n$, and $T(t)u(0) = \sum_{n\geq 1} e^{-n^2 t} u^n(0) f^n, t \geq 0$. Since

$$L(u_s) = \sum_{n\geq 1} < L(u_s), f^n > f^n,$$

it follows that

$$T(t-s)L(u_s) = \sum_{n\geq 1} e^{-n^2(t-s)} < L(u_s), f^n > f^n.$$

From the definition of L, it follows that

$$< L(u_s), f^n > = < b\int_{-\infty}^{0} u(s+\theta)\, d\theta, f^n > = b\int_{-\infty}^{s} < u(r), f^n > \, dr.$$

Hence $u^n(t)$ satisfies the equation

$$u^n(t) = e^{-n^2 t} u^n(0) + \int_0^t e^{-n^2(t-s)} b \int_{-\infty}^{s} u^n(r)\, dr\, ds.$$

Taking the derivatives successively, we know that $u^n(t)$ satisfies the equation

$$x'(t) = -n^2 x(t) + b\int_{-\infty}^{t} x(r) dr$$

with the initial condition $x(\theta) = \phi^n(\theta) :=< \phi(\theta), f^n >, \theta \in (-\infty, 0]$, and the equation $x''(t) = -n^2 x'(t) + bx(t)$ with the initial conditions

$$x(0) = \phi^n(0), \quad x'(0) = -n^2\phi^n(0) + b\int_{-\infty}^{0} \phi^n(\theta)d\theta.$$

Set $\lambda_{\pm}^n = (-n^2 \pm \sqrt{D_n})/2, D_n = n^4 + 4b$. If $D_n \neq 0$, the solution is given as

$$\begin{aligned} u^n(t) &= x(t) \\ &= \frac{-\lambda_-^n x(0) + x'(0)}{\sqrt{D_n}} e^{\lambda_+^n t} + \frac{\lambda_+^n x(0) - x'(0)}{\sqrt{D_n}} e^{\lambda_-^n t}. \\ &= \frac{(n^2 + \sqrt{D_n})e^{\lambda_-^n t} - (n^2 - \sqrt{D_n})e^{\lambda_+^n t}}{2\sqrt{D_n}} \phi^n(0) \\ &\quad + \frac{(e^{\lambda_+^n t} - e^{\lambda_-^n t})b}{\sqrt{D_n}} \int_{-\infty}^{0} \phi^n(\theta)\, d\theta. \end{aligned}$$

If $D_n = 0$, then $\lambda_{\pm}^n = -n^2/2$, and the solution is given as

$$\begin{aligned} u^n(t) &= x(t) \\ &= \left[(1 + (n^2 t/2))x(0) + tx'(0)\right] e^{-n^2 t/2} \\ &= \left[(1 - (n^2 t/2))\phi^n(0) + bt\int_{-\infty}^{0} \phi^n(\theta)\, d\theta\right] e^{-n^2 t/2}. \end{aligned}$$

Since $b < 0$, it follows that $\Re\lambda_{\pm}^n < 0$ for $n \geq 1$, and there exists a constant c_1 such that, for $D_n \neq 0$,

$$|u^n(t)| \leq c_1 \left[|\phi^n(0)| + \frac{|b|}{\sqrt{|D_n|}} \int_{-\infty}^{0} |\phi^n(\theta)|\, d\theta\right],$$

and for $D_n = 0$,

$$|u^n(t)| \leq c_1 \left[|\phi^n(0)| + |b| \int_{-\infty}^{0} |\phi^n(\theta)|\, d\theta\right].$$

Since

$$\int_{-\infty}^{0} |\phi^n(\theta)|\, d\theta \leq \int_{-\infty}^{0} |\phi(\theta)|\, d\theta$$

for $n \geq 1$, it follows that

$$\begin{aligned} \left(\sum_{n=1}^{\infty} |u^n(t)|^2\right)^{1/2} &\leq c_1 \left(\sum_{n=1}^{\infty} |\phi^n(0)|^2\right)^{1/2} \\ &\quad + c_1|b| \left(1 + \sum_{n^4+4b\neq 0} \frac{1}{|D_n|}\right)^{1/2} \int_{-\infty}^{0} |\phi(\theta)|\, d\theta. \end{aligned}$$

for $t \geq 0$. Thus there exists a constant M such that

$$|u(t,\phi)| \leq M\|\phi\| \quad \text{for } t \geq 0, \phi \in \mathcal{B}.$$

Take an N such that $D_n > 0$ for $n \geq N$. For $n \geq N$, set

$$h_n = (n^2 + \sqrt{D_n})/2\sqrt{D_n}, \quad k_n = (n^2 - \sqrt{D_n})/2\sqrt{D_n}.$$

Since $e^{\lambda_-^n t} \leq e^{-t}, e^{\lambda_+^n t} \leq 1$, it follows that

$$|u^n(t)| \leq (|h_n|e^{-t} + |k_n|)|\phi^n(0)| + \frac{2|b|}{\sqrt{D_n}} \int_{-\infty}^0 |\phi(\theta)|\, d\theta.$$

Set $H_m = \sup\{|h_n| : n \geq m\}, K_m = \sup\{|k_n| : n \geq m\}, m \geq N$. Since $\lim_{n\to\infty} |h_n| = 1$, and since $\lim_{n\to\infty} k_n = 0$, H_m is bounded, $\lim_{m\to\infty} K_m = 0$, and

$$\begin{aligned}
\left(\sum_{n\geq m} |u^n(t)|^2\right)^{1/2} &\leq (H_m e^{-t} + K_m)\left(\sum_{n\geq m} |\phi^n(0)|^2\right)^{1/2} \\
&\quad + \left(\sum_{n\geq m} \frac{4|b|^2}{D_n}\right)^{1/2} \int_{-\infty}^0 |\phi(\theta)|\, d\theta \\
&\leq \left(H_m e^{-t} + K_m + \left(\sum_{n\geq m} \frac{4|b|^2}{D_n}\right)^{1/2}\right)\|\phi\|.
\end{aligned}$$

Let $\epsilon > 0$. Then there exist $m = m(\epsilon) \geq N$ and $\tau_1(\epsilon)$ such that

$$\left(\sum_{n\geq m} |u^n(t)|^2\right)^{1/2} < \epsilon\|\phi\| \quad \text{for } t > \tau_1(\epsilon), \phi \in \mathcal{B}.$$

Since $|u^n(t,\phi)| \to 0$ as $t \to \infty$ uniformly for $1 \leq n < m$ and for ϕ such that $\|\phi\| \leq 1$, there exists a $\tau_2(\epsilon) > 0$ such that $(\sum_{n<m} |u^n(t)|^2)^{1/2} < \epsilon$ provided $t \geq \tau_2(\epsilon), \|\phi\| \leq 1$. Consequently, it follows that, if $t > \tau_3(\epsilon) := \max\{\tau_1(\epsilon), \tau_2(\epsilon)\}$ and if $\|\phi\| \leq 1$, then $|u(t,\phi)| < \sqrt{2}\epsilon$. Since $u(t,\phi)$ is linear in ϕ, it follows that $|u(t,\phi)| \leq \epsilon\|\phi\|$ for $t > \tau(\epsilon) := \tau_3(\epsilon/\sqrt{2}), \phi \in \mathcal{B}$.

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